The Reynolds-Averaged Navier-Stokes Equations: A detailed derivation

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***Abstract*—The subject of Reynolds-Averaged Navier-Stokes Equations is treated on many books dealing with micro-meteorology and related subjects. Often tersely, after space demands. In this work, aimed at junior undergraduate audience, the author presents an as full as possible derivation of RANS equations while minimally resorting to assumptions.**

I. Introduction

This work presents one derivation, of the many possible, of Reynolds-Averaged Navier-Stokes equations (RANS). This is not new material, by any way. But in my perception this widely treated subject is in ways addressed to professional scientists rather than undergraduate level (I'm thinking to my cubs-of-engineer in particular).

The common literature presentation of RANS assumes readers to fill-in the passages as an exercise, and this approach has surely some value for future physicists and mathematicians. But it might be something too much for people who hsn't a vested interest or a passion on the subject of micro-meteorology.

This, the original motivation.

Then working the passages myself (waiting nothing special) I realized something deeper was lurking there. As a friend of mine (a physicist) loves to say, the Devil hides among details, and I'm beginning to suspect he's right. What's sure some of these details are more "interesting", than I have expected. Sometimes in themselves. Other times in a somewhat more surprising way.

Surely the RANS look very mathematical, and in this they are similar to many other formulae we can encounter on a physics book. Working them out, you realize you could not derive them from the original Navier-Stokes equations only mathematically. Some of the passages, spotted here and there, are intrinsically *physical*. They entail some form of simplification, of human assumptions about the reality around us (quite literally in case of our Planetary Boundary Layer - PBL for friends).

And this in turn clarifies one thing: as a simplification of reality, the RANS are ("just"?) a *model*. A mathematical one, valid if precise conditions are fulfilled.

It also makes us add a tiny bread to the question "Why is the World so mathematical?", so common among my physicist friends. When an (almost) little girl I've always been quite suspicious of it. As a (sort of - more on later - "mathematician") I took for granted mathematics is a sort of distillate of neural connections and structures forged by mutation and natural selection eons ago, much before the human kind emerged as something visible. Mathematics is effective as a description of reality because the underlying neural structures also are (we wouldn't be here to write or read otherwise). That sort of awareness was also central in my own personal epistemology: Nature is a large collection of subjects we could interact with, and who can tell stories; our understanding is greatly improved, if we use some *lingua franca*, and maths surely is a good canididate.

I told "sort of mathematician", and I confirm. Around late Jurassic I got a degree in applied maths. But before of that I enjoyed a voluntary internship in ecosystems ecology (at the University of Milan - my supervisors were very kind accepting and welcoming me when still ih high school after my sciences professor's presentation letter: I was *really* a little girl, passionate of Nature and much more interested in mountain hikes and bycicle rides than in the nerdlike stuff like, among them, studying: surely something quite far from the type of the genius). After my degree I worked as an ersatz-engineer and, later even, sort-of physicist of the lower atmosphere after my employer discovered (!?) I was a woman after all and induced me to find another job somewhere else (they considered directing a programmable-systems-for-safety-critical-applications testing group a for-men-only job - pfui, *amateurs*). So I can happily say I'm an ersatz of an ersatz-something, and as one of my professors convincingly explained this is more than sufficient to qualify me as *not* a Real Mathematician (Real Mathematicians, in his view, do not eat, breath, drink, have sex, contemplate flowers, etc, etc: they only do Mathematics, and are perfectly satisfied of that). (For completeness, I've to admit I'm proud of this non-definition.)

This long boring self-introduction is my way of saying *read this all at your own risk*. You should do better if you assume all I write is to be taken with pliers. OK, I wrote some peer-reviewed papers (recently under my name after I got the courage to, and time ago under a (sort of!) more authoritatively looking art pseudonym), but what I really am is semi-skilled worker in the field of measurement and control systems mostly for the physics of lower atmosphere.

(My passion for Nature and (to my high school maths teacher dismal) personal epistemology have remained unchanged.)

Before we proceed, and to the benefit of people unaccustomed to it, let me informally introduce the Einstein index notation: following people who preceded me I'll use it extensively., and so, better to be prepared. It's not difficult, and a real lifesaver for a lazy person once you get accustomed to it. It has to do with indices in expressions. Let's make some practical example.

Suppose we're dealing with the wind vector (we'll do soon extensively). We could write it as $\mathbf{v} = [v_x, v_y, v_z]$ if we choose an appropriate reference system. Or, in my new micro-meteorologists community, $\mathbf{v} = [u, v, w]$. But nothing prevents we introduce an index and write $\mathbf{v} = [u_1, u_2, u_3]$ instead. Or even, just u_i, with the tacit assumption $i = 1, 2, 3$. In the expression u_i the index is not repeated, and according to Einstein's convention this means it should be expanded in individual components 1, 2 and 3.

I've shown one index, but nothing prevents to use two or more. In case of two non-repeating indices, an expression like α_{ij} represents a 3-by-3 matrix: no repetitions equals expansion, that is $\alpha_{11}, \alpha_{12}, \ldots, \alpha_{22}, \alpha_{23}, \ldots, \alpha_{33}$.

Now, let's *repeat* an index, for example α_{jj}. This time, the meaning is *summation*: $\alpha_{11} + \alpha_{22} + \alpha_{33}$. Or, if you prefer the way we (sort-of) mathematicians love using, $\sum_{j=1}^{3} \alpha_{jj}$.

I could have also written α_{ii}, and the meaning would have been the same under Einstein's convention: the sum $\alpha_{11} + \alpha_{22} + \alpha_{33}$. But I will not: as micro-meteorologists do, I'll consistently reserve i for the non-repeated index (the "equation number") and j for the repeated. We'll soon meet some creative and tricky use of Einstein's notation, in addition to the inoffensive: I'll explain them in all cases.

This said, we can really begin.

II. The Navier-Stokes equations in their most usual form

In boundary layer meteorology airflow is represented by the instantaneous, Cartesian-reference Navier-Stokes equations written using Einstein's index conventions:

$$\frac{\partial u_i}{\partial t} + u_j \frac{\partial u_i}{\partial x_j} = -\delta_{i3} g + f_c \varepsilon_{ij3} u_j - \frac{1}{\rho} \frac{\partial p}{\partial x_i} + \nu_j \frac{\partial^2 u_i}{\partial x_j^2} \quad (1)$$

where u_i is the i-th component of wind speed ($u_1 = u$,$u_2 = v$,$u_3 = w$), t the time, x_i the i-th coordinate ($x_1 = x$,$x_2 = y$,$x_3 = z$), δ_{ij} the Kronecker symbol ($\delta_{ij} = 0$ if $i \neq j$, $= 1$ if $i = j$), g the gravity acceleration, $f_c = 2\omega \sin\phi$ the Coriolis parameter, ω the Earth revolution speed, ϕ the latitude, ρ the air pressure, ε_{ijk} the three-indices Levi-Civita symbol (more on in a following section) and ν_j the diffusivity coefficient along the i-th axis (with the additional assumption $\nu_1 = \nu_2 = \nu_3$.

III. Solutions to Navier-Stokes equations

In principle, given appropriate (and infinitely detailed) initial and boundary conditions, equations (1) could be solved yielding an instantaneous wind field

$$u_i = u_i(x, y, z, t) \quad (2)$$

On paper, all rights, if not for the little detail that finding it is one of the renown Millennium Prize Problems (in the same league of telling whether $P = NP$. and other similar headaches; incidentally, would you find it, please tell me in advance so we can share the one million dollar). (In the real World we content ourselves with numerical approximations: they demand a lot of computing work, but are satisfactory enough for applications - and computers are not prone to complaining of

There is an issue, however: solutions to Navier-Stokes equations are maybe too rich, as they contain the (interesting) overall *en-masse* airflow along with other parts usually considered less attracting like for example turbulent eddies.

IV. Isolating the en-masse flow

In order to isolate the *en-masse* airflow, it is advisable to define what is not an *en-masse* flow. In an ideal World this could be made on purely physical, phenomenological grounds, by decomposing the airflow in its components, each coming from one specific and objectively discernible phenomenon, like turbulent eddies I've mentioned in the previous section.

A problem with this approach is that to date the full list of physical phenomena occurring in air under natural conditions is still incomplete. We know of turbulent eddies, and also coherent structures and meandering (with no attempt from my side to order them in some rational ranking).

So, given the impossibility for the moment to establish a full theory of natural airflow we have to resort to some empirical definition. The one in current use is unfortunately rooted in mathematics, not physics, and takes the form of an operational definition: the "anything but *en-masse* flow" is characterized by a "zero mean". To date we have nothing better.

In more precise terms, given any quantity we could imagine to measure in the airflow (a velocity component, or temperature, or the concentration of some gas, or whatever), and let it be s, can be decomposed in the sum of a *mean part*, $\overline{s}$, and a *fluctuation* s', the latter having zero mean, that is, $\overline{s'} = 0$.

Of course, $s = s(t)$ is a function (also) of time, and the same is of $\overline{s} = \overline{s}(t)$ and $s' = s'(t)$.

This approach is known under the name *Reynolds decomposition*, as the first quantitative description of turbulence was devised in the 19th century by the English physicist Osborne Reynolds. The idea was not new when professor Reynolds gave it a precise form, however, and its beginning can be traced to Leonardo da Vinci who in his *Codex Atlanticus* drafted clearly the concept, introducing the word *turbulence* for the first time incidentally.

Would we identify one way to give mathematical substance to the Reynolds decomposition, then we could apply it to the Navier-Stokes equations, so obtaining an averaged form of them accounting only for the *en-masse* flow.

V. Reynolds mean

When Osborne Reynolds first established his decomposition relation he intended the "mean" as something quite precisely defined: an *ensemble average*. This can be formulated in the conceptual frame of a laboratory experiment in which many replicas

$$s_i = s_i(t) \quad (3)$$

are collected ($i = 1, \ldots, n$). Then their average is constructed as

$$\overline{s} = \frac{1}{n} \sum_{i=1}^{n} s_i(t) \quad (4)$$

and the fluctuations associated to each replica can then be constructed by difference:

$$s'_i = s_i - \overline{s} \tag{5}$$

From this construction it is clear that

$$\overline{s'_i} = 0 \tag{6}$$

Property (6) is not the only one: from the definition of ensemble average we could also get other immediate properties (r, s are two signals, α, β two real numbers):

$$\overline{\alpha r + \beta s} = \alpha\overline{r} + \beta\overline{s} \tag{7}$$

$$\overline{r\overline{s}} = \overline{\overline{r}s} = \overline{r} \cdot \overline{s} \tag{8}$$

Properties (6), (7) and (8) are immediate consequences of the definition of ensemble average, but may also be assumed as a requirement list for any "mean". They are useful: their presence allows manipulating instantaneous equations to obtain their averaged counterparts - we'll see this soon.

Before to proceed, however, it is worth observing the decision of using properties (6), (7) and (8) as a (partial) definition of "mean" is quite delicate mathematically. Its original framing is that of a lab experiment executed with many replicas. But in Nature we can't set initial and boundary conditions for an "experiment": there is no laboratory!

We still can do something, but in order to we have to make some assumptions. First of all, we can say s is a realization (possibly the only one available) of an underlying random process S. Among users the common assumption is made S is ergodic: in this case average over "replicas" (multiple realizations) can be replaced by time averages.

Many concrete definitions of "time averages" can be introduced. One (widely used) example is the "block average", defined in a later section. Properties (6), (7) and (8) can then be used to weed out "means" which are not acceptable, at least formally. One such case is the so-called "McMillen mean", defined on a discrete signal $s = \{s_i\}_{i=1}^{n}$ by the AR(1) relation

$$\overline{s}_1 = s_1 \quad \overline{s}_k = \beta s_k + (1-\beta)\overline{s}_{k-1} \tag{9}$$

We'll say about that later.

VI. Axioms for "means", tentative definition

It is anyway worth observing relations (6), (7) and (8), known in the community as "Reynolds postulates", do not form a complete system of axioms. The author impression, however, is we got close to it.

As a first attempt to establish a family of axioms for means, it is first useful to delimit a sort of natural environment for them. A first element of it is a domain of acceptable signals - and here physics reappears somewhat. A *signal* is a function $s : \mathbb{R} \mapsto \mathbb{R}$ subject to the following conditions:

1) $|s(t)| < \infty$.
2) $s(t)$ is continuously differentiable up to some $h > 0$.

Both conditions are *physical* in their essence. The first states changes occur within a finite band, and then the energy involved (whatever its form) is also finite. The second tells that whatever steep and apparently unpredictable these changes are, they will never involve discontinuities, cusps, or other cases in which the derivative does not exist.

As the definition of signal is stated, its domain of definition coincides with $\mathbb{R}$ (we can identify it with time). But no generality is lost if we assume $s : \mathbb{R}^m \mapsto \mathbb{R}$ with any positive integer m. And indeed, in sake of physical interpretation a value of $m = 4$ may be taken for granted (one dimension represents "time", and the other three the "spatial" dimensions).

Condition 1 may in this case be maintained as is. Condition 2 can be reformulated by stating $\frac{\partial^k s}{\partial\alpha^{k_1}\partial\alpha^{k_2}\cdots\partial\alpha^{k_m}}$ exists continuous for any positive integer $k \leq h$ ($k_1+k_2+\cdots+k_m = k$).

Then we set $\mathbb{S}$ as the set of all signals satisfying conditions 1 and 2, equipped with the operations of sum of two signals

$$(r + s) = (r + s)(t) = r(t) + s(t) \tag{10}$$

the product of two signals

$$(rs) = (rs)(t) = r(t)s(t) \tag{11}$$

and the product of a signal by a scalar

$$(\alpha s)(t) = \alpha s(t) \tag{12}$$

With these positions, it is clear $\mathbb{S}$ is (also) a vector space. We now consider a *functional* ϕ on $\mathbb{S}$ as a generic mapping from $\mathbb{S}$ to $\mathbb{S}$ that is an operator associating a signal to a signal.

We say the functional ϕ is *involutory* if

$$\phi(\phi(s)) = \phi(s) \tag{13}$$

whenever $s \in \mathbb{S}$. A *projector* of $\mathbb{S}$ on a subspace $\mathbb{T}$ of $\mathbb{S}$ is a functional ϕ for which $\phi(\mathbb{S}) \subseteq \mathbb{T}$.

A functional ϕ is *linear* if

$$\phi(\alpha r + \beta s) = \alpha\phi(r) + \beta\phi(s) \tag{14}$$

A functional ϕ is *multiplicative* if

$$\phi(r\phi(s)) = \phi(\phi(r)s) = \phi(r)\phi(s) \tag{15}$$

A *generalized mean* over $\mathbb{S}$ (or just *mean* for short) is an involutory linear multiplicative functional on $\mathbb{S}$. A (generalized) mean is often indicated as $\overline{s}$ using an overline instead of the functional notation $\phi(s)$, but this by no means should be considered mandatory. In fact these two notations will be interchanged as convenience suggests.

VII. Residuals

The *residual* of signal s associated to a functional ϕ is the functional

$$\rho(s) = s - \phi(s) \tag{16}$$

For historical reasons, the residual of a generalized mean is named *fluctuation*, and we write

$$s' = s - \overline{s} \tag{17}$$

If a functional ϕ is linear, then its residual also is. To see this, let us consider $\rho(\alpha r + \beta s)$. According to the definition we have

$$\rho(\alpha r + \beta s) = \alpha r + \beta s - \phi(\alpha r + \beta s) \tag{18}$$

By the linearity of ϕ the right member of (18) can be written as

$$\alpha r + \beta s - \alpha\phi(r) + \beta\phi(s) = \alpha\left[r - \phi(r)\right] + \beta\left[s - \phi(s)\right] \tag{19}$$

that is, $\alpha\rho(r) + \beta\rho(s)$ as desired.

What happens to the residual, if its functional is multiplicative? Let's consider

$$\rho(\phi(r\phi(s))) = \phi(r\phi(s)) - \phi(\phi(r\phi(s))) \tag{20}$$

The right member's leftmost term can be written as $\phi(r)\phi(s)$ by multiplicativity. And the rightmost? It is $\phi(\phi(r)\phi(s))$. So, we can say

$$\rho(\phi(r\phi(s))) = \phi(r)\phi(s) - \phi(\phi(r)\phi(s)) \tag{21}$$

that is, $\rho(\phi(r\phi(s))) = \rho(\phi(r)\phi(s))$. With almost identical steps we can say $\rho(\phi(\phi(r)s)) = \rho(\phi(r)\phi(s))$, and we see ρ is also multiplicative.

At this point it would be tempting to imagine the residual of an involutory functional is also involutory. And in a sense it is, but in a much stronger sense. In this case we would get

$$\rho(\phi(s)) = \phi(s) - \phi(\phi(s)) = \phi(s) - \phi(s) = 0 \tag{22}$$

This is quite an important result: if a functional is involutory its residual is zero. Can we say the reverse is true? That is, if the residual is zero then the associated operator is involutory? The answer is in general negative. All we could say is:

$$\rho(s) = s - \phi(s) = 0 \tag{23}$$

or

$$s = \phi(s) \tag{24}$$

But this is a property of the *signal*, not the operator. Formula (24) can in fact be seen as a *functional equation*. Its solutions (which need neither to exist nor be unique) is a *fixed point* of functional ϕ. Quite an unusual "point" indeed, as it is an entire signal, but names remain often attached for historical reason regardless their possibly misleading value...

We could say more would ϕ be linear in addition to involutory: in this case,

$$\phi(\rho(s)) = \phi(s - \phi(s)) = \phi(s) - \phi(\phi(s)) = \rho(\phi(s)) = 0 \tag{25}$$

This is quite interesting. The fact of being linear and involutory is all what we need to say the functional of the residual being zero, a result identical to (6). The fact of being multiplicative plays no special role in this case.

On the positive side we can say this proves our "axioms" of generalized means are equivalent to Reynolds postulates (6) to (8).

In section V the promise was made to show the AR(1) filter (9) is not a mean. Now we can see: the AR(1) filter is not involutory, and this is enough to exclude it from the set of generalized means.

VIII. Derivatives as functionals

In section VI the requirement of being differentiable with continuity was made for signals, up to some appropriately "high" order k. We can change our perspective now, and consider time or directional derivatives as special cases of functionals. Indeed they are, being able to associate an element of $\mathbb{S}$ to another element of $\mathbb{S}$.

Differentiation rules allow us to say something more. In view of this objective we may use generic first order partial derivatives, which will be indicated as $\frac{\partial}{\partial a}$ regardless what a actually designates, time or a spatial direction. From calculus we know

$$\frac{\partial}{\partial a}(\alpha r + \beta s) = \alpha\frac{\partial}{\partial a}r + \beta\frac{\partial}{\partial a}s \tag{26}$$

That's quite like the discovery of hot water, but partial derivative is a linear operator.

The partial derivative is not involutory, however:

$$\frac{\partial}{\partial a}\left(\frac{\partial}{\partial a}s\right) = \frac{\partial^2}{\partial a^2}s \neq \frac{\partial}{\partial a}s \tag{27}$$

Interestingly (and also trivially, if you remember calculus) the partial differential functional has a fixed point:

$$\frac{\partial}{\partial a}s = s \implies s = \exp(a) \tag{28}$$

May we say something about multiplicativity? Let's see:

$$\frac{\partial}{\partial a}\left(r\frac{\partial}{\partial a}s\right) = \frac{\partial}{\partial a}r\frac{\partial}{\partial a}s + r\frac{\partial}{\partial a}\frac{\partial}{\partial a}s = \frac{\partial}{\partial a}r\frac{\partial}{\partial a}s + r\frac{\partial^2}{\partial a^2}s \tag{29}$$

$$\frac{\partial}{\partial a}\left(\left(\frac{\partial}{\partial a}r\right)s\right) = \frac{\partial}{\partial a}r\frac{\partial}{\partial a}s + s\frac{\partial}{\partial a}\frac{\partial}{\partial a}r = \frac{\partial}{\partial a}r\frac{\partial}{\partial a}s + s\frac{\partial^2}{\partial a^2}r \tag{30}$$

The two expressions differ, and do not coincide with $\frac{\partial}{\partial a}r\frac{\partial}{\partial a}s$: the differentiation functional is not multiplicative.

Nothing new, nor unexpected: the differentiation functional is quite far from a generalized mean, as it should be. More important to our cases, is whether differentiation is well-behaved with respect to generalized means.

In particular, it would be interesting to discover the behavior of differentiation with respect to a functional ϕ. We start by writing the definition of the differentiation operator:

$$\frac{\partial}{\partial a}s(a) = \lim_{h\to 0}\frac{s(a + h \cdot j_a) - s(a)}{h} \tag{31}$$

where $a = [x, y, z, t]$ is the position and instant at which the signal is evaluated and j_a its corresponding unit vector. By our assumptions the derivative exists. It makes then sense to wrap it into the functional ϕ, thus obtaining

$$\phi\left(\frac{\partial}{\partial a}s\right) = \phi\left(\lim_{h\to 0}\frac{s(a+h\cdot j_a)-s(a)}{h}\right) \tag{32}$$

In line of principle it is not guaranteed ϕ can be moved inside the limit sign: in addition to the axioms we preliminarily did set in section VI we also need ϕ to be continuous, in the sense $\phi(s(a))$ is a continuous function of a whatever the signal s. This is indeed not that difficult to state, as the "means" we could imagine are continuous in this sense. But then we may write

$$\phi\left(\lim_{h\to 0}\frac{s(a+h\cdot j_a)-s(a)}{h}\right) = \lim_{h\to 0}\phi\left(\frac{s(a+h\cdot j_a)-s(a)}{h}\right) \tag{33}$$

If ϕ is linear we can also say

$$\lim_{h\to 0}\phi\left(\frac{s(a+h\cdot j_a)-s(a)}{h}\right) = \lim_{h\to 0}\frac{\phi\left(s(a+h\cdot j_a)\right)-\phi\left(s(a)\right)}{h} \tag{34}$$

Linearity and continuity then suffice to prove the chain of equalities

$$\phi\left(\frac{\partial}{\partial a}s\right) = \frac{\partial}{\partial a}\phi(s) \tag{35}$$

which in case of generalized means may be written as

$$\overline{\frac{\partial}{\partial a}s} = \frac{\partial}{\partial a}\overline{s} \tag{36}$$

Relation (36) may be iterated, and so extended to derivatives from the second on, until the maximum k assumed for signals.

IX. Sectioning the Navier-Stokes equations: the total derivative

We are now in the position to express the Reynolds averaging of Navier-Stokes equations. In sake of notational simplicity it is better to divide the equation (1) in parts, then expand them from Einstein to standard convention.

The first building block of (1) we'll consider is the total derivative of wind,

$$\frac{\partial u_i}{\partial t} + u_j\frac{\partial u_i}{\partial x_j} \tag{37}$$

which upon averaging becomes

$$\overline{\frac{\partial u_i}{\partial t} + u_j\frac{\partial u_i}{\partial x_j}} \tag{38}$$

By linearity, the preceding equations become

$$\overline{\frac{\partial u_i}{\partial t}} + \overline{u_j\frac{\partial u_i}{\partial x_j}} \tag{39}$$

The leftmost term may be expressed directly as

$$\frac{\partial \overline{u_i}}{\partial t} \tag{40}$$

To express the rightmost we apply directly the Reynolds decomposition:

$$\overline{\left(\overline{u_j}+u_j'\right)\frac{\partial\left(\overline{u_i}+u_i'\right)}{\partial x_j}} \tag{41}$$

or

$$\overline{\overline{u_j}\frac{\partial \overline{u_i}}{\partial x_j} + \overline{u_j}\frac{\partial u_i'}{\partial x_j} + u_j'\frac{\partial \overline{u_i}}{\partial x_j} + u_j'\frac{\partial u_i'}{\partial x_j}} \tag{42}$$

By linearity, the preceding formula becomes

$$\overline{\overline{u_j}\frac{\partial \overline{u_i}}{\partial x_j}} + \overline{\overline{u_j}\frac{\partial u_i'}{\partial x_j}} + \overline{u_j'\frac{\partial \overline{u_i}}{\partial x_j}} + \overline{u_j'\frac{\partial u_i'}{\partial x_j}} \tag{43}$$

Now we use the fact a generalized mean is multiplicative

$$\overline{\overline{u_j}}\,\overline{\frac{\partial \overline{u_i}}{\partial x_j}} + \overline{\overline{u_j}}\,\overline{\frac{\partial u_i'}{\partial x_j}} + \overline{u_j'}\,\overline{\frac{\partial \overline{u_i}}{\partial x_j}} + \overline{u_j'\frac{\partial u_i'}{\partial x_j}} \tag{44}$$

and involutory:

$$\overline{u_j}\,\overline{\frac{\partial \overline{u_i}}{\partial x_j}} + \overline{u_j}\,\overline{\frac{\partial u_i'}{\partial x_j}} + \overline{u_j'}\,\overline{\frac{\partial \overline{u_i}}{\partial x_j}} + \overline{u_j'\frac{\partial u_i'}{\partial x_j}} \tag{45}$$

Now we use formula (36) for derivatives, so obtaining

$$\overline{u_j}\frac{\partial \overline{\overline{u_i}}}{\partial x_j} + \overline{u_j}\frac{\partial \overline{u_i'}}{\partial x_j} + \overline{u_j'}\frac{\partial \overline{\overline{u_i}}}{\partial x_j} + \overline{u_j'\frac{\partial u_i'}{\partial x_j}} \tag{46}$$

We then use involutority again:

$$\overline{u_j}\frac{\partial \overline{u_i}}{\partial x_j} + \overline{u_j}\frac{\partial \overline{u_i'}}{\partial x_j} + \overline{u_j'}\frac{\partial \overline{u_i}}{\partial x_j} + \overline{u_j'\frac{\partial u_i'}{\partial x_j}} \tag{47}$$

As we have seen in section VII, involutorily and linearity imply $\overline{s'} = 0$. Using this property we may finally say

$$\overline{u_j}\frac{\partial \overline{u_i}}{\partial x_j} + \overline{u_j}\frac{\partial 0}{\partial x_j} + 0\frac{\partial \overline{u_i}}{\partial x_j} + \overline{u_j'\frac{\partial u_i'}{\partial x_j}} = \overline{u_j}\frac{\partial \overline{u_i}}{\partial x_j} + \overline{u_j'\frac{\partial u_i'}{\partial x_j}} \tag{48}$$

The latter formula comes always to me, the author, as a surprise. It's unexpected. One would imagine, if intuition is the only guide, the mean of otal derivative to be just $\overline{u_j'}\frac{\partial u_i'}{\partial x_j}$. But no: that's also that little disturbing term, $\overline{u_j'\frac{\partial u_i'}{\partial x_j}}$. On a purely formal side, it's something survived to the Reynolds averaging "axioms". But, what does it mean exactly?

I anticipate the remaining part of the reasoning is grounded on physics, not math: this departs from the calculus-only steps we've performed until now. And, as often happens, it adds something interesting and diverse to the discourse. We have no space or time to get deeper in phylosophical matters, but as we'll see within seconds the derivation of Reynolds Averaged Navier-Stokes equation *cannot occur only by mathematical means*. It gives birth to one of the many physical models having a mathematical form, but it is by no means "mathematics". Not strictly. It is something, if you like, with a beautiful mathematical dress, but with living flesh inside. Rooted in our World, our reality as it is. One more reason to not astonish the physical World is so "mathematical". Maybe it isn't, strictly speaking. Maybe, maths is a language.

Its "poetry" very well fit the real World, but in a sense maybe a true mathematician would not appreciate.

The first step, by the way, is mathematical. We notice that, by high-school differentiation-of-a-product rule,

$$\frac{\partial u_i' u_j'}{\partial x_j} = u_i' \frac{\partial u_j'}{\partial x_j} + u_j' \frac{\partial u_i'}{\partial x_j} \quad (49)$$

or, if you prefer,

$$u_j' \frac{\partial u_i'}{\partial x_j} = \frac{\partial u_i' u_j'}{\partial x_j} - u_i' \frac{\partial u_j'}{\partial x_j} \quad (50)$$

If we apply the generalized mean to both terms, we see

$$\overline{u_j' \frac{\partial u_i'}{\partial x_j}} = \overline{\frac{\partial u_i' u_j'}{\partial x_j} - u_i' \frac{\partial u_j'}{\partial x_j}} \quad (51)$$

Now we apply linearity, and get the rightmost member as

$$\overline{\frac{\partial u_i' u_j'}{\partial x_j}} - \overline{u_i' \frac{\partial u_j'}{\partial x_j}} \quad (52)$$

Now, a mathematical coincidence with an interesting physical consequence: in $u_i' \frac{\partial u_j'}{\partial x_j}$, the $\frac{\partial u_j'}{\partial x_j}$ factor occurs with the index j repeated: according to Einstein's convention, its meaning is really

$$\frac{\partial u_j'}{\partial x_j} = \frac{\partial u'}{\partial x} + \frac{\partial v'}{\partial y} + \frac{\partial w'}{\partial z} \quad (53)$$

It has a name: in the lingo of vector people it is the divergence of the fluctuations of airflow. And here comes the physical factlet: it is the fluctuation part of the divergence *of the whole airflow*,

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} \quad (54)$$

which if wind speed is much less than the speed of sound in still air, *is negligible*. Its mean also is, together with the mean of its fluctuation. That given, formula (52) may be rewritten this way:

$$\overline{\frac{\partial u_i' u_j'}{\partial x_j}} - \overline{u_i' \frac{\partial u_j'}{\partial x_j}} = \overline{\frac{\partial u_i' u_j'}{\partial x_j}} - \overline{u_i' 0} = \overline{\frac{\partial u_i' u_j'}{\partial x_j}} \quad (55)$$

In conclusion, we (*physically*!) proved

$$u_j' \frac{\partial u_i'}{\partial x_j} = \frac{\partial u_i' u_j'}{\partial x_j} \quad (56)$$

and then, in turn,

$$\overline{\frac{\partial u_i}{\partial t} + u_j \frac{\partial u_i}{\partial x_j}} = \frac{\partial \overline{u_i}}{\partial t} + \frac{\partial \overline{u_i' u_j'}}{\partial x_j} \quad (57)$$

It is now worth noting $\overline{u_i' u_j'}$ has a statistical interpretation: it is the *covariance* (in generalized terms) of u_i and u_j.

X. The gravity term

Nothing to say here, this term is constant and remains unchanged upon application of the generalized mean. The only tricky thing is notational: the term reads $-\delta_{i3} g$, where δ_{ij} is the discrete analogous of the Dirac delta, equal to 1 if $i = j$, 0 otherwise. But i in this case is the *equation* index: written as it is, the gravity term is zero on the first two equations of the three, namely the ones for u and v. In case of the third equation, for w, the term assumes its value $-g$.

XI. The Coriolis term

The term $f_c \varepsilon_{ij3} u_j$ in (1) represents the effect of Coriolis apparent force, and is a direct consequence of having chosen a reference frame fixed to the Earth surface, notoriously non-inertial. Of f_c we have seen already in section II: to any respect we can consider it a constant (it depends just on position through latitude, and we know position is fixed).

Again however we have a bit of Einstein' sorcery: ε_{ijk} is Levi-Civita symbol, equal to 1 if ijk form an even permutation of 1,2,3; -1 if ijk form an odd permutation; and 0 otherwise. In our case, we have the symbol particularized to ε_{ij3}, with j repeating on u_j and then indicating a sum. We expand it, considering i designates the equation: for $i = 1$, the only non-zero value of ε_{1j3} is found when $j = 2$, and is +1 (123 form an even permutation of 1,2,3) so the Coriolis term assumes the form

$$+f_c u_2 \quad (58)$$

For $i = 2$ the only non-zero value ε_{2j3} occurs when $j = 1$ and is -1 (213 is an odd permutation of 1,2,3). The Coriolis term takes then the form

$$-f_c u_1 \quad (59)$$

Last, if $i = 3$ we have ε_{3j3}. The index 3 is repeated, so whatever the value of j, $3j3$ is neither an even nor an odd permutation: then, $\varepsilon_{3j3} = 0$. This makes a lot of sense: Coriolis force has no vertical component.

XII. The pressure term

Term $-\frac{1}{\rho} \frac{\partial p}{\partial x_i}$ in (1) represents the effect of pressure. In Planetary Boundary Layer physics pressure plays a much smaller role than in, say, synoptic meteorology: on a small spatial scale pressure changes are typically quite small. Nevertheless, pressure and air density are subject to fluctuation and their averaging needs then be performed.

Air density ρ occurs on denominator, and this is a case we never met, nor some axiom exists regarding it. So we cannot in general say $\overline{\frac{1}{\rho}} = \frac{1}{\overline{\rho}}$. But on the other side we can apply Taylor expansion and write

$$\frac{1}{\overline{\rho} + \rho'} = \frac{1}{\overline{\rho}} - \frac{\rho'}{\overline{\rho}^2} + \cdots + (-1)^k \left(\frac{\rho'}{\overline{\rho}} \right)^k \frac{1}{\overline{\rho}} + \cdots \quad (60)$$

Taylor series sometimes are good sources of simple and useful approximations: when luck assists truncated versions of them may be either used to compute the desired approximand to some prescribed accuracy (in this case by fixing in advance an order to the polynomial representing the truncated series) or to extend accuracy adding more terms. Examples of well-behaved Taylor series are $\exp x$, $\sin x$ and $\cos x$. But unfortunately vanilla-like behavior cannot be taken for granted, and indeed it is quite rare: in this respect it helps when the mean around which variations are taken is large compared to variation itself: this ensures individual terms decay more quickly to zero as their order increases, which in turn permits a faster convergence of approximations. If, in addition, it is guaranteed the series terms have alternating signs (our case!) then the error we get by truncating the series has a magnitude smaller than the first neglected term (another calculus rule, this time from freshman university year).

In micromteorology books the approximation

$$\frac{1}{\overline{\rho}+\rho'} \approx \frac{1}{\overline{\rho}} \tag{61}$$

is commonly made. The author's proposal is to retain one more term:

$$\frac{1}{\overline{\rho}+\rho'} \approx \frac{1}{\overline{\rho}} - \frac{\rho'}{\overline{\rho}^2} = \frac{1}{\overline{\rho}}\left(1-\frac{\rho'}{\overline{\rho}}\right) \tag{62}$$

If we agree on this, we may write the pressure term in approximate form

$$-\left(\frac{1}{\overline{\rho}} - \frac{\rho'}{\overline{\rho}^2}\right)\frac{\partial\left(\overline{p}+p'\right)}{\partial x_i} \tag{63}$$

then apply linearity of derivatives and a little algebra to get

$$-\frac{1}{\overline{\rho}}\frac{\partial\overline{p}}{\partial x_i} - \frac{1}{\overline{\rho}}\frac{\partial p'}{\partial x_i} + \frac{\rho'}{\overline{\rho}^2}\frac{\partial\overline{p}}{\partial x_i} + \frac{\rho'}{\overline{\rho}^2}\frac{\partial p'}{\partial x_i} \tag{64}$$

Now we apply averaging and take its linearity into account:

$$-\overline{\frac{1}{\overline{\rho}}\frac{\partial\overline{p}}{\partial x_i}} - \overline{\frac{1}{\overline{\rho}}\frac{\partial p'}{\partial x_i}} + \overline{\frac{\rho'}{\overline{\rho}^2}\frac{\partial\overline{p}}{\partial x_i}} + \overline{\frac{\rho'}{\overline{\rho}^2}\frac{\partial p'}{\partial x_i}} \tag{65}$$

Now, we can apply multiplicativity. By it, the first term of (65) becomes

$$-\overline{\frac{1}{\overline{\rho}}}\,\overline{\frac{\partial\overline{p}}{\partial x_i}} \tag{66}$$

What about $\overline{\frac{1}{\overline{\rho}}}$? Still, from a mathematical standpoint we should carry the generalized mean oversign inside, but *physically* we can observe that in the Planetary Boundary Layer, adopting SI units, the value of $\overline{\rho}$ is represented by quite a large number (about 335 hPa on tip of Mt Everest), subject to small changes in the hourly-or-less time scale typical of PBL phenomena. Then we can bona fide assume the ratio $\frac{1}{\overline{\rho}}$ *is* a generalized mean. Then, by involutorily,

$$\overline{\frac{1}{\overline{\rho}}} = \frac{1}{\overline{\overline{\rho}}} = \frac{1}{\overline{\rho}} \tag{67}$$

The other part of the first term is quite inoffensive: we just apply the property of mean derivatives,

$$\overline{\frac{\partial\overline{p}}{\partial x_i}} = \frac{\partial\overline{\overline{p}}}{\partial x_i} \tag{68}$$

and then involutority:

$$\frac{\partial\overline{\overline{p}}}{\partial x_i} = \frac{\partial\overline{p}}{\partial x_i} \tag{69}$$

so we can write the first term as intuition could suggest:

$$-\frac{1}{\overline{\rho}}\frac{\partial\overline{p}}{\partial x_i} \tag{70}$$

We now apply multiplicativity to second and third terms:

$$-\overline{\frac{1}{\overline{\rho}}\frac{\partial p'}{\partial x_i}} + \overline{\frac{\rho'}{\overline{\rho}^2}\frac{\partial\overline{p}}{\partial x_i}} = -\overline{\frac{1}{\overline{\rho}}}\,\overline{\frac{\partial p'}{\partial x_i}} + \overline{\frac{\rho'}{\overline{\rho}^2}}\,\overline{\frac{\partial\overline{p}}{\partial x_i}} \tag{71}$$

By using the same reasoning as with the first term we could wed out reciprocals of $\overline{\rho}$ and stay happy: they will not interfere. More importantly, we can write

$$\overline{\frac{\partial p'}{\partial x_i}} = \frac{\partial\overline{p'}}{\partial x_i} = \frac{\partial 0}{\partial x_i} = 0 \tag{72}$$

and

$$\overline{\frac{\rho'}{\overline{\rho}^2}} = \frac{\overline{\rho'}}{\overline{\rho}^2} = \frac{0}{\overline{\rho}^2} = 0 \tag{73}$$

so we can write

$$-\overline{\frac{1}{\overline{\rho}}\frac{\partial p'}{\partial x_i}} + \overline{\frac{\rho'}{\overline{\rho}^2}\frac{\partial\overline{p}}{\partial x_i}} = 0 \tag{74}$$

The fourth term involves a covariance, as we have seen for momentum in section IX. As in that case we can not decompose it any further, but by moving the mean density outside (which involves linearity, multiplicativity and the considerations on ρ already exposed in this section:

$$\overline{\frac{\rho'}{\overline{\rho}^2}\frac{\partial p'}{\partial x_i}} = \frac{1}{\overline{\rho}^2}\overline{\rho'\frac{\partial p'}{\partial x_i}} \tag{75}$$

All we have done allows to finally write the pressure term as

$$-\frac{1}{\rho}\frac{\partial p}{\partial x_i} = -\frac{1}{\overline{\rho}}\frac{\partial\overline{p}}{\partial x_i} + \frac{1}{\overline{\rho}^2}\overline{\rho'\frac{\partial p'}{\partial x_i}} \tag{76}$$

XIII. The diffusive term

The last term in Navier-Stokes equations accounts for diffusion:

$$\nu_j\frac{\partial^2 u_i}{\partial x_j^2} \tag{77}$$

When averaging it, we may assume the diffusion coefficient $\nu = \nu_i$ to be constant: upon application of linearity and the formula for averaging derivatives we straightforwardly get

$$\nu_j\frac{\partial^2\overline{u_i}}{\partial x_j^2} \tag{78}$$

XIV. The RANS, finally

By assembling the various averaged terms we have written so far it is possible to see the (Patrizia' style) RANS:

$$\frac{\partial \overline{u_i}}{\partial t}+\frac{\partial \overline{u_i' u_j'}}{\partial x_j}=-\delta_{i3}g+f_c\varepsilon_{ij3}\overline{u_j}-\frac{1}{\overline{\rho}}\frac{\partial \overline{p}}{\partial x_i}+\frac{1}{\overline{\rho}^2}\overline{\rho'\frac{\partial p'}{\partial x_i}}+\nu_j\frac{\partial^2 \overline{u_i}}{\partial x_j^2} \quad (79)$$

We can also see the "standard" RANS, by just dropping the covariance from the pressure term:

$$\frac{\partial \overline{u_i}}{\partial t}+\frac{\partial \overline{u_i' u_j'}}{\partial x_j}=-\delta_{i3}g+f_c\varepsilon_{ij3}\overline{u_j}-\frac{1}{\overline{\rho}}\frac{\partial \overline{p}}{\partial x_i}+\nu_j\frac{\partial^2 \overline{u_i}}{\partial x_j^2} \quad (80)$$

XV. Going further

A cursory examination of the RANS (80) shows they are formally more intricate than their instantaneous counterparts (1): they also contain "strange" momentum covariance terms; this is even more evident with Patrizia's version, where in addition to momentum there are also pressure-density covariances.

Would we try solving the RANS numerically, we should also say something of these covariances: in any regard they can be seen as *additional* independent variables. That's a dire problem: the original NS are three (nonlinear) equations in three unknowns, and might in principle be "solved". Not so for the RANS, unless of course we either add many more equations to "close" the system, or we say something about them.

The approach followed by the scientific community has been to express the covariances in terms of simpler variables: still additional, but less in number and less intricate. This approach is named "closure", and account of it may be found in any classical text dealing with micro-meteorology.

I will refrain doing so, however: my objective was quite didactical in nature, and surely more limited. Dealing also with closures would just have been a duplication of existing excellent works: an unneeded waste of bytes, paper and neurons.

It is however interesting, if you like, to go further. In this view, and without attempting to endorse any author, I would suggest the classical book by R. Stull, *An Introduction to Boundary Layer Meteorology*, Kluwer Academic Publishing, 1988: quite dated, yet still very useful and, as far as RANS are involved, actual (besides, professor Stull was one of the authors active in inventing clever closure relationships). The book is still available for sale.

For the Italian reader, a quark more recent (and advanced) alternative is R. Sozzi, T. Georgiadis, M. Valentini, *Introduzione alla turbolenza atmosferica - Concetti, stime, misure,* Pitagora Editrice, 2002. As far as the author knows, this title is not any longer available for sale, but can be easily found in Italian faculty libraries (I'm one of the lucky people having their own personal copy, but don't even try asking it even for lending: I'll defent it to life).

Both Stull and Sozzi et al books are very informative about the formulae, but saying they have an introductory character would be a positive falsity. To get a grasp of boundary layer meteorology I'd recommend S. Pal Arya, *Introduction to Micrometeorology*, Academic Press, 2001.

Apart from some elementary functional analysis lingo, anything mathematical in this work could not demand more than high-school and fresh(wo)man level undergraduate material: I did my best to stay "simple".

Of course you may have still doubts: as I mentioned at the very beginning of this work this is a really healthy attitude. In case, you may find at my e-mail address, patti.favaron@gmail.com: I cannot promise I'll answer in real time, but I'll do my best to stay in contact. If you write me, however, please specify the title of this report, just to understand where your question is coming from.

I wish you peace and prosperity. And maybe, you are even perceiving my Vulcan salute, mr.Spock style.